

Upper bounds for centerlines*

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Abstract

In 2008, Bukh, Matoušek, and Nivasch conjectured that for every n -point set S in \mathbb{R}^d and every k , $0 \leq k \leq d-1$, there exists a k -flat f in \mathbb{R}^d (a “centerflat”) that lies at “depth” $(k+1)n/(k+d+1) - O(1)$ in S , in the sense that every halfspace that contains f contains at least that many points of S . This claim is true and tight for $k=0$ (this is Rado’s centerpoint theorem), as well as for $k=d-1$ (trivial). Bukh et al. showed the existence of a $(d-2)$ -flat at depth $(d-1)n/(2d-1) - O(1)$ (the case $k=d-2$).

In this paper we concentrate on the case $k=1$ (the case of “centerlines”), in which the conjectured value for the leading constant is $2/(d+2)$. We prove that $2/(d+2)$ is an *upper bound* for the leading constant. Specifically, we show that for every fixed d and every n there exists an n -point set in \mathbb{R}^d for which no line in \mathbb{R}^d lies at depth greater than $2n/(d+2) + o(n)$. This point set is the “stretched grid”—a set which has been previously used by Bukh et al. for other related purposes.

Hence, in particular, the conjecture is now settled for \mathbb{R}^3 .

Keywords: centerpoint, centerline, centerflat, stair-convexity, stretched grid.

1 Introduction

Given a finite set $S \subset \mathbb{R}^d$ and a point $x \in \mathbb{R}^d$, define the *depth* of x in S as the minimum of $|\gamma \cap S|$ over all closed halfspaces γ that contain x . Rado’s centerpoint theorem (1947, [9]) states that for every n -point set $S \subset \mathbb{R}^d$ there exists a point $x \in \mathbb{R}^d$ at depth at least $n/(d+1)$ in S . Such a point x is called a *centerpoint*.

Centerpoints, besides being a basic notion in discrete geometry, have also been studied in connection with statistical data analysis: The centerpoint x is a single point that describes, in some sense, a given “data set” S [4, 6, 10].

The notion of *depth* that we use in this paper is sometimes called *halfspace depth* or *Tukey depth*, to distinguish it from other notions of depth (see, for example, [7]).

The constant $1/(d+1)$ in the centerpoint theorem is easily shown to be tight: Take $d+1$ affinely independent points in \mathbb{R}^d , and let S be obtained by replacing each of these

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points by a tiny “cloud” of $n/(d+1)$ points. Then no point in \mathbb{R}^d lies at depth greater than $n/(d+1)$ in S .

In this paper we consider a generalization of the centerpoint theorem in which the desired object is not a deep point, but rather a deep k -flat for some given $0 \leq k < d$. Thus, let us define the *depth* of a k -flat $f \subset \mathbb{R}^d$ in S as the minimum of $|\gamma \cap S|$ over all closed halfspaces γ that contain f .

Bukh, Matoušek, and Nivasch [2] proved that for every n -point set $S \subset \mathbb{R}^d$ there exists a $(d-2)$ -flat $f \subset \mathbb{R}^d$ at depth at least $(d-1)n/(2d-1) - O(1)$.¹

It is trivial that there always exists a $(d-1)$ -flat at depth at least $n/2$ in S . In [2] it was conjectured that, in general, for every k , $0 \leq k \leq d-1$, there exists a k -flat at depth at least $(k+1)n/(k+d+1) - O(1)$ in S , and that the fraction $(k+1)/(k+d+1)$ is sharp. Such a flat would be called a *centerflat*; and we call this conjecture the *centerflat conjecture*.

The centerflat conjecture is closely related to the *center transversal theorem* of Dol’nikov [5] and Živaljević and Vrećica [11]; it states that, if $S_1, \dots, S_{k+1} \subset \mathbb{R}^d$ are point sets of sizes n_1, \dots, n_{k+1} , respectively, then there exists a k -flat $f \subset \mathbb{R}^d$ that simultaneously lies at depth at least $n_i/(d-k+1)$ in each S_i .

As far as we know, however, the centerflat conjecture itself has not been studied until very recently. Arocha et al. [1] have obtained a lower bound of $1/(d-k+1)$ for the leading constant in the conjecture (see Corollary 3 there). The same constant can also be obtained using the center transversal theorem, by setting all the sets S_i to S . Actually, one can obtain this constant much more simply, by projecting S into \mathbb{R}^{d-k} and then applying the centerpoint theorem. However, the conjectured constant of $(k+1)/(k+d+1)$ is larger than $1/(d-k+1)$ for all $1 \leq k \leq d-2$.

In this paper we focus on the case $k=1$ of the centerflat conjecture (the case of “centerlines”). For this case the conjecture predicts a value of $2/(d+2)$ for the leading constant, and we show that this value cannot be improved. Specifically:

Theorem 1.1. *Let $d \geq 2$ be fixed. Then, for every n there exists an n -point set $G_s \subset \mathbb{R}^d$ such that for every line $\ell \subset \mathbb{R}^d$ there exists a halfspace containing ℓ and containing at most $2n/(d+2) + o(n)$ points of G_s .*

Combining Theorem 1.1 with the above-mentioned result in [2], we conclude that in \mathbb{R}^3 there is always a line at depth $2n/5 - O(1)$, and that the fraction $2/5$ is sharp.

The set G_s in the theorem is the “stretched grid”—a point set that was previously used by Bukh et al. [3] for obtaining lower bounds for weak ϵ -nets, upper bounds for the so-called *first selection lemma*, and for other related purposes (see also [8]). Unfortunately, we have been unable to find a simple, “cloud”-based construction for proving Theorem 1.1, like the construction mentioned above for centerpoints.

¹They showed that there exist $2d-1$ hyperplanes passing through a common $(d-2)$ -flat that partition S into $4d-2$ parts, each of size at least $n/(4d-2) - O(1)$. This $(d-2)$ -flat is the desired f , since every halfspace that contains it must completely contain at least $2d-2$ of the parts.

2 The stretched grid and stair-convexity

The *stretched grid* is an axis-parallel grid of points where, in each direction i , $2 \leq i \leq d$, the spacing between consecutive “layers” increases rapidly, and furthermore, the rate of increase for direction i is much larger than that for direction $i - 1$. To simplify calculations, we will also make the coordinates increase rapidly in the first direction.²

The formal definition is as follows: Given n , the desired number of points, let $m = n^{1/d}$ be the side of the grid (assume for simplicity that this quantity is an integer), and let

$$G_s = \{(K_1^{a_1}, K_2^{a_2}, \dots, K_d^{a_d}) : a_i \in \{0, \dots, m-1\} \text{ for all } 1 \leq i \leq d\}, \quad (1)$$

for some appropriately chosen constants $1 < K_1 \ll K_2 \ll K_3 \ll \dots \ll K_d$. Each constant K_i must be chosen appropriately large in terms of K_{i-1} and in terms of m . We choose the constants as follows:

$$K_1 = 2d, \quad K_2 = K_1^m, \quad K_3 = K_2^m, \quad \dots, \quad K_d = K_{d-1}^m. \quad (2)$$

Throughout this paper we refer to the d -th coordinate as the “height”, so a hyperplane in \mathbb{R}^d is *horizontal* if all its points have the same last coordinate; and a line in \mathbb{R}^d is *vertical* if all its points share the first $d - 1$ coordinates. A *vertical projection onto* \mathbb{R}^{d-1} is obtained by removing the last coordinate. The i -th *horizontal layer* of G_s is the subset of G_s obtained by letting $a_d = i$ in (1).

The following lemma is not actually used in the paper, but it provides the motivation for the stretched grid:

Lemma 2.1. *Let $a \in G_s$ be a point at horizontal layer 0, and let $b \in G_s$ be a point at horizontal layer i . Let c be the point of intersection between segment ab and the horizontal hyperplane containing layer $i - 1$. Then $|c_i - a_i| \leq 1$ for every $1 \leq i \leq d - 1$.*

Lemma 2.1 follows from a simple calculation (we chose the constants K_i in (2) large enough to make this and later calculations work out).

The grid G_s is hard to visualize, so we apply to it a logarithmic mapping π that converts G_s into the uniform grid in the unit cube.

Formally, let $\text{BB} = [1, K_1^{m-1}] \times \dots \times [1, K_d^{m-1}]$ be the bounding box of the stretched grid, let $[0, 1]^d$ be the unit cube in \mathbb{R}^d , and define the mapping $\pi: \text{BB} \rightarrow [0, 1]^d$ by

$$\pi(x) = \left(\frac{\log_{K_1} x_1}{m-1}, \dots, \frac{\log_{K_d} x_d}{m-1} \right).$$

Then, it is clear that $\pi(G_s)$ is the uniform grid in $[0, 1]^d$.

We say that two points $a, b \in \text{BB}$ are *c-close in coordinate i* if the i -th coordinates of $\pi(a)$ and $\pi(b)$ differ by at most $c/(m-1)$. Roughly speaking, this means that a and b are at most i layers apart in the i -th direction. Otherwise, we say that a and b are *c-far in coordinate i* . Two points are *c-close* if they are *c-close* in every coordinate, and they are *c-far* if they are *c-far* in every coordinate.

²The most natural way to define the stretched grid is using the notion of *infinitesimals* from non-standard analysis. But we avoid doing so in order to keep the exposition accessible.

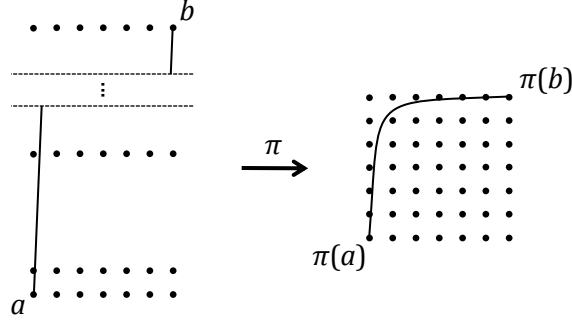


Figure 1: The stretched grid and the mapping π in the plane. The stretched grid is too tall to be drawn entirely, so an intermediate portion of it has been omitted. A line segment connecting two points is also shown, as well as its image under π . (The first coordinate of the stretched grid does not increase geometrically in this picture.)

Lemma 2.1 implies that the map π transforms straight-line segments into curves composed of almost-straight axis-parallel parts: Let s be a straight-line segment connecting two points of G_s . Then $\pi(s)$ ascends almost vertically from the lower endpoint, almost reaching the height of the higher endpoint, before moving significantly in any other direction; from there, it proceeds by induction. See Figure 1.

This observation motivates the notions of *stair-convexity*, which describe, in a sense, the limit behavior of π as $m \rightarrow \infty$.

2.1 Stair-convexity

We recall a few notions from [3].

Given a pair of points $a, b \in \mathbb{R}^d$, the *stair-path* $\sigma(a, b)$ between them is a polygonal path connecting a and b and consisting of at most d closed line segments, each parallel to one of the coordinate axes. The definition goes by induction on d ; for $d = 1$, $\sigma(a, b)$ is simply the segment ab . For $d \geq 2$, after possibly interchanging a and b , let us assume $a_d \leq b_d$. We set $a' = (a_1, \dots, a_{d-1}, b_d)$, and we let $\sigma(a, b)$ be the union of the segment aa' and the stair-path $\sigma(a', b)$; for the latter we use the recursive definition, ignoring the common last coordinate of a' and b .

Note that, if c and d are points along $\sigma(a, b)$, then $\sigma(c, d)$ coincides with the portion of $\sigma(a, b)$ that lies between c and d .

A set $X \subseteq \mathbb{R}^d$ is said to be *stair-convex* if for every $a, b \in X$ we have $\sigma(a, b) \subseteq X$.

Given a set $X \subset \mathbb{R}^d$ and a real number h , let $X(h)$ (the *horizontal slice at height* h) be the vertical projection of $\{x \in X : x_d = h\}$ into \mathbb{R}^{d-1} . In [3] it was shown that a set $X \subset \mathbb{R}^d$ is stair-convex if and only if the following two conditions hold: (1) every horizontal slice $X(h)$ is stair-convex; (2) for every $h_1 \leq h_2 \leq h_3$ such that $X(h_3) \neq \emptyset$ we have $X(h_1) \subseteq X(h_2)$ (meaning, the horizontal slice can only grow with increasing height, except that it can end by disappearing abruptly).³ For convenience we call this

³This criterion was stated slightly incorrectly in [3]; the formulation given above is the correct one.

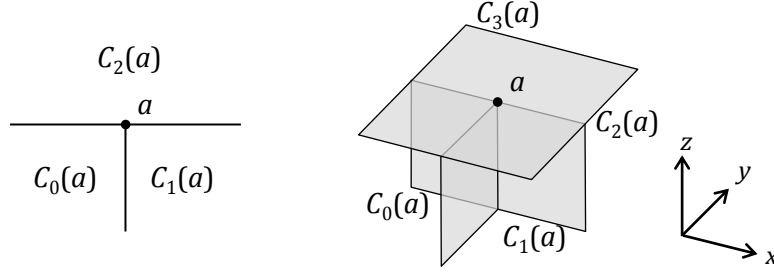


Figure 2: Components with respect to a point a in the plane (left) and in \mathbb{R}^3 (right).

criterion *monotonicity of slices*.

Let $a \in \mathbb{R}^d$ be a fixed point, and let $b \in \mathbb{R}^d$ be another point. We say that b has *type 0 with respect to a* if $b_i \leq a_i$ for every $1 \leq i \leq d$. For $1 \leq j \leq d$ we say that b has *type j with respect to a* if $b_j \geq a_j$ but $b_i \leq a_i$ for every i satisfying $j + 1 \leq i \leq d$. (It might happen that b has more than one type with respect to a , but only if some of the above inequalities are equalities.)

Given a point $a \in \mathbb{R}^d$, let $C_i(a)$ (the *i -th component with respect to a*) be the set of all points in \mathbb{R}^d that have type i with respect to a . Thus,

$$\begin{aligned} C_0(a) &= (-\infty, a_1] \times \cdots \times (-\infty, a_d]; \\ C_i(a) &= (-\infty, \infty)^{i-1} \times [a_i, \infty) \times (-\infty, a_{i+1}] \times \cdots \times (-\infty, a_d], \quad \text{for } 1 \leq i \leq d. \end{aligned}$$

See Figure 2.

We now introduce a new notion, that of a *stair-halfspace*. Stair-halfspaces are, roughly speaking, the stair-convex analogue of Euclidean halfspaces.

Definition 2.2: Let $a \in \mathbb{R}^d$ be a point, and let $\emptyset \subsetneq I \subsetneq \{0, \dots, d\}$ be a set of indices. Then the set $\bigcup_{i \in I} C_i(a)$ is called a *stair-halfspace*, and a is its *vertex*.

Lemma 2.3. *Let H be a stair-halfspace. Then both H and $\mathbb{R} \setminus H$ are stair-convex.*

Proof. Consider H . Every horizontal slice $H(h)$ of H is either empty, all of \mathbb{R}^{d-1} , or a $(d-1)$ -dimensional stair-halfspace. Thus, by induction, $H(h)$ is always stair-convex. Furthermore, for every $h_1 \leq h_2 \leq h_3$ such that $H(h_3) \neq \emptyset$ we have $H(h_1) \subseteq H(h_2)$. Thus, H is stair-convex by monotonicity of slices. A similar argument applies for $\mathbb{R} \setminus H$. \square

(There are other sets in \mathbb{R}^d that deserve to be called *stair-halfspaces*, that do not fit into the above definition; for example, the set $\{(x, y) \in \mathbb{R}^2 : x \geq 0\}$. But Definition 2.2 covers all the stair-halfspaces that we will need in this paper.)

Two stair-halfspaces $\bigcup_{i \in I} C_i(a)$ and $\bigcup_{i \in I} C_i(b)$ with the same index set I are said to be *combinatorially equivalent*.

Note that the map π preserves stair-convexity notions (since it operates component-wise and is monotone in each component). In particular, let $X \subseteq \text{BB}$; then: (1) X is

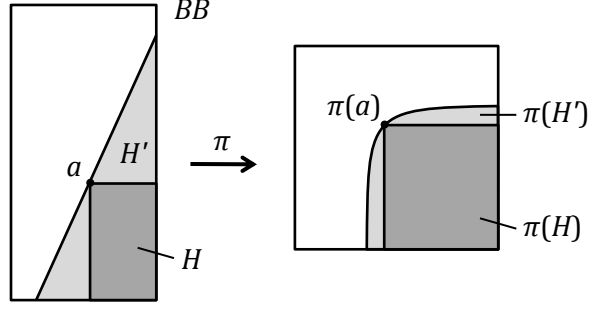


Figure 3: For every stair-halfspace H there exists a Euclidean halfspace H' that closely approximates H within BB : The images of H and H' under π almost coincide. (The figure is not to scale.)

stair-convex if and only if $\pi(X)$ is stair-convex; (2) X is a stair-path if and only if $\pi(X)$ is a stair-path; (3) there exists a stair-halfspace H such that $X = H \cap BB$ if and only if there exists a combinatorially equivalent stair-halfspace H' such that $\pi(X) = H' \cap [0, 1]^d$.

The following lemma shows that every stair-halfspace is, in a sense, the limit of the image under π of a Euclidean halfspace.

Lemma 2.4. *Let $a \in BB$ be a point, and let $H = \bigcup_{i \in I} C_i(a)$ be a stair-halfspace with vertex a and index set $\emptyset \subsetneq I \subsetneq \{0, \dots, d\}$. Then there exists a Euclidean halfspace H' with $a \in \partial H'$ such that, for every point $x \in BB$ that is 1-far from a , we have $x \in H$ if and only if $x \in H'$. (See Figure 3.)*

Proof. The desired Euclidean halfspace is

$$H' = \{x \in \mathbb{R}^d : s_0 + s_1 \frac{x_1}{a_1} + s_2 \frac{x_2}{a_2} + \dots + s_d \frac{x_d}{a_d} \geq 0\},$$

where the s_i 's are small signed integers chosen to satisfy the following conditions:

1. For every $0 \leq i \leq d$, s_i is positive if $i \in I$, and negative otherwise.
2. $\sum_i s_i = 0$.
3. We have $1 \leq |s_i| \leq d$ for all i .

Such a choice is always possible since $1 \leq |I| \leq d$, so there will be both positive and negative s_i 's.

The second condition above ensures that a lies on the boundary of H' .

Now consider a point $x \in BB$ that is 1-far from a . Thus, we have either $x_i \geq K_i a_i$ or $x_i \leq a_i / K_i$ for every coordinate i .

Let i be the largest coordinate such that $x_i \geq K_i a_i$, if it exists. Consider the sum

$$s_0 + s_1 \frac{x_1}{a_1} + s_2 \frac{x_2}{a_2} + \dots + s_d \frac{x_d}{a_d}. \quad (3)$$

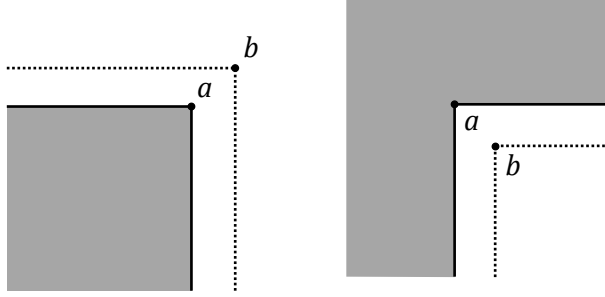


Figure 4: Translating a stair-halfspace outwards.

We claim that the term $s_i \frac{x_i}{a_i}$ is larger in absolute value than all the other terms in (3) combined. Indeed, for $j < i$ we have

$$|s_j| \frac{x_j}{a_j} \leq |s_j| x_j \leq |s_j| K_j^{m-1} = |s_j| \frac{K_{j+1}}{K_j} \leq |s_j| \frac{K_i}{K_j} \leq |s_j| \frac{x_i}{a_i K_j} \leq d |s_i| \frac{x_i}{a_i K_j} \leq 2^{-j} |s_i| \frac{x_i}{a_i}.$$

This is because the constants K_i were chosen appropriately large in (2). Similarly, for $j > i$ we have $|s_j| \frac{x_j}{a_j} \leq d/K_j \leq 2^{-j}$.

Thus, the sign of (3) is the sign of s_i , which implies that $x \in H'$ if and only if $i \in I$.

If, on the other hand, $x_i \leq a_i/K_i$ for all i , then, by a similar argument, the sign of (3) is the sign of s_0 , so $x \in H'$ if and only if $0 \in I$. \square

The following lemma formalizes what we mean by translating a stair-halfspace “outwards”:

Lemma 2.5. *Let $H = \bigcup_{i \in I} C_i(a)$ be a stair-halfspace with vertex $a \in \mathbb{R}^d$ and index set $\emptyset \subsetneq I \subsetneq \{0, \dots, d\}$. Let $b \in \mathbb{R}^d$ be another point such that, for each $1 \leq i \leq d$, we have $b_i < a_i$ if $i \in I$, and $b_i > a_i$ otherwise.*

Let $H' = \bigcup_{i \in I} C_i(b)$ be the stair-halfspace combinatorially equivalent to H with vertex b . Then $H \subset H'$. (See Figure 4.)

Proof. Let $p \in C_i(a)$ for some $i \in I$. We have $p_i \geq a_i > b_i$ if $i \geq 1$, and $p_j \leq a_j$ for each $i+1 \leq j \leq d$. We need to show that $p \in C_k(b)$ for some $k \in I$.

Let k be the largest index such that $p \in C_k(b)$. Then $p_k \geq b_k$ if $k \geq 1$, and $p_j < b_j$ for each $k+1 \leq j \leq d$.

If $k = i$ then $k \in I$ and we are done. Otherwise, we must have $k > i$, or else we would have $p_i < b_i < a_i \leq p_i$. But $k > i$ implies that $b_k \leq p_k \leq a_k$. Since $b_k \neq a_k$, we have $b_k < a_k$, which implies $k \in I$, as desired. \square

3 Proof of Theorem 1.1

In this section we prove that the stretched grid G_s satisfies Theorem 1.1.

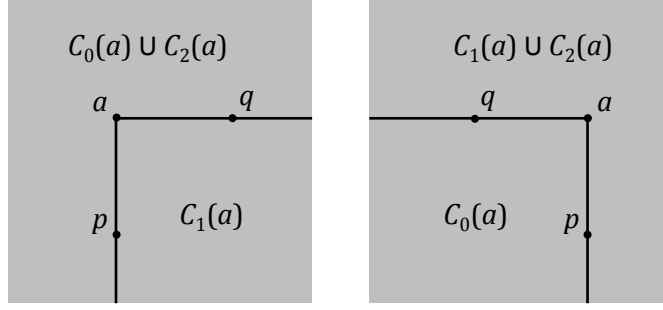


Figure 5: The base case of the covering lemma.

Lemma 3.1 (Covering lemma). *Let p, q be two points in \mathbb{R}^d , $d \geq 2$. Then there exists a family \mathcal{H} of $(d-1)(d+2)/2$ stair-halfspaces, each one containing both p and q , such that the stair-halfspaces of \mathcal{H} together cover \mathbb{R}^d exactly $d-1$ times (apart from the points on the boundary of the stair-halfspaces of \mathcal{H} , which might be covered more times).*

Proof. We proceed by induction on d . For the base case $d = 2$, we want to construct two stair-halfplanes containing p and q that cover \mathbb{R}^2 exactly once. Suppose without loss of generality that $q_2 \geq p_2$. Let $a = (p_1, q_2)$. If $p_1 \leq q_1$, then let $\mathcal{H} = \{C_0(a) \cup C_2(a), C_1(a)\}$; otherwise, let $\mathcal{H} = \{C_0(a), C_1(a) \cup C_2(a)\}$ (see Figure 5).

Now assume $d \geq 3$. Let \bar{p} and \bar{q} denote the vertical projection of p and q into \mathbb{R}^{d-1} , respectively. By induction, let \mathcal{H}' be a family of $(d-2)(d+1)/2$ stair-halfspaces in \mathbb{R}^{d-1} containing \bar{p} and \bar{q} and covering \mathbb{R}^{d-1} exactly $d-2$ times.

Suppose without loss of generality that $p_d \leq q_d$. As a first step, construct the family of stair-halfspaces in \mathbb{R}^d

$$\mathcal{H}^* = \{H \times (-\infty, q_d] : H \in \mathcal{H}'\}.$$

This corresponds to adding q_d as the d -th coordinate to the vertex of every stair-halfspace $H \in \mathcal{H}'$, and then reinterpreting the components of H as being d -dimensional.

Note that $|\mathcal{H}^*| = (d-2)(d+1)/2$, that p and q belong to every stair-halfspace in \mathcal{H}^* , and that \mathcal{H}^* covers the *lower part* of \mathbb{R}^d (namely, $\mathbb{R}^{d-1} \times (-\infty, q_d]$) exactly $d-2$ times, and it does not cover the *upper part* of \mathbb{R}^d (namely, $\mathbb{R}^{d-1} \times [q_d, \infty)$) at all.

Next, let m be an index $0 \leq m \leq d-1$ such that $\bar{q} \in C_m(\bar{p})$ (e.g., let $m \leq d-1$ be the largest index for which $p_m \leq q_m$, or zero if no such index exists). Let $a = (p_1, \dots, p_{d-1}, q_d)$, and define the family of stair-halfspaces

$$\mathcal{H}^{**} = \{C_i(a) \cup C_d(a) : 0 \leq i \leq d-1, i \neq m\} \cup \{C_m(a)\},$$

all having a as vertex.

First, note that the stair-halfspaces of \mathcal{H}^{**} cover the lower part of \mathbb{R}^d exactly once (because each component $C_0(a), \dots, C_{d-1}(a)$ is present exactly once), and they cover the upper part of \mathbb{R}^d exactly $d-1$ times (because the component $C_d(a)$ is present $d-1$ times).

Furthermore, note that each stair-halfspace of \mathcal{H}^{**} contains both p and q : The components $C_i(a)$, $i \leq d-1$ contain p ; the component $C_d(a)$ contains q ; and the component $C_m(a)$ contains both p and q , by the choice of m .

Thus, the desired family of stair-halfspaces is $\mathcal{H} = \mathcal{H}^* \cup \mathcal{H}^{**}$: It contains

$$\frac{(d-2)(d+1)}{2} + d = \frac{(d-1)(d+2)}{2}$$

stair-halfspaces, and it covers \mathbb{R}^d exactly $d-1$ times. \square

Remark 3.2: The points p and q actually lie on the *boundary* of each stair-halfspace of \mathcal{H} . This can be seen by recursively characterizing the boundary of a stair-halfspace (a “stair-hyperplane”), and using induction.

Proof of Theorem 1.1. Let G_s be the n -point stretched grid in \mathbb{R}^d , let BB be its bounding box, and let ℓ be a line in \mathbb{R}^d . We want to construct a Euclidean halfspace that contains ℓ and contains at most $2n/(d+2) + o(n)$ points of G_s .

If ℓ does not intersect the interior of BB then there is nothing to do. Otherwise, let p' and q' be the intersection points of ℓ with the boundary of BB, and let $p = \pi(p')$, $q = \pi(q')$ be the corresponding points in the boundary of $[0, 1]^d$.

Let \mathcal{H} be the family of stair-halfspaces guaranteed by Lemma 3.1 for the points p and q . By the pigeonhole principle, there must exist a stair-halfspace $H \in \mathcal{H}$ such that $\text{vol}(H \cap [0, 1]^d) \leq 2/(d+2)$. Move the vertex a of H “outwards” by distance $1/(m-1)$ in each direction, so that p and q are still contained in H and are far enough from its boundary (recall Lemma 2.5). This increases the volume of $H \cap [0, 1]^d$ by only $o(1)$.

Now, the volume of $H \cap [0, 1]^d$ closely approximates the fraction of points of $\pi(G_s)$ contained in H ; specifically, $\text{vol}(H \cap [0, 1]^d) = |\pi(G_s) \cap H|/n \pm O(n^{(d-1)/d})$. This is because $H \cap [0, 1]^d$ is the union of a constant number of axis-parallel boxes, and the claim is clearly true for axis-parallel boxes.

Let $a' = \pi^{-1}(a)$, and let H' be the stair-halfspace combinatorially equivalent to H having a' as its vertex. Then, $|G_s \cap H'| = |\pi(G_s) \cap H| \leq 2n/(d+2) + o(n)$. Furthermore, we have $p', q' \in H'$, and in fact, p' and q' are 1-far from a' . Therefore, the Euclidean halfspace H'' promised by Lemma 2.4 contains both p and q , and, like H' , it contains at most $2n/(d+2) + o(n)$ points of G_s .

It might still be possible that H'' does not contain all of ℓ , but this is easy to fix: The sets ℓ and $\text{BB} \setminus H''$ are disjoint, and they are both convex. Therefore, there exists a hyperplane h that separates them. Let H''' be the halfspace bounded by h that contains ℓ . Then $H''' \cap \text{BB} \subseteq H'' \cap \text{BB}$, so H''' can only contain *fewer* points of G_s than H'' . \square

4 Generalization to k -flats

We conjecture that the stretched grid G_s in fact gives a tight upper bound of $(k+1)/(k+d+1)$ for all k for the leading constant in the centerflat conjecture.

We have a sketch of a proof. Its main ingredients are: (1) an appropriate definition of *stair- k -flats*, the stair-convex equivalent of Euclidean k -flats; and (2) a generalization

of Lemma 3.1 to the effect that, for every stair- k -flat $f \subset \mathbb{R}^d$, there exists a family \mathcal{H} of $\binom{d-1}{k} \frac{d+k+1}{k+1}$ stair-halfspaces, each one containing f in its boundary, and together covering \mathbb{R}^d exactly $\binom{d-1}{k}$ times.

However, we have some problems formalizing the argument: We have been unable to rigorously prove that our stair-flats are indeed the “limit case under π ” of Euclidean k -flats, and we have also been unable to deal with some degenerate stair-flats.

For the interested reader, In Appendix A we spell out the argument, pointing out the “holes” that we still have.

Acknowledgments. Thanks to Jiří Matoušek for some insightful conversations on this and related topics, to Roman Karasev for some useful correspondence, and to the anonymous referee for giving a careful review and providing extensive comments.

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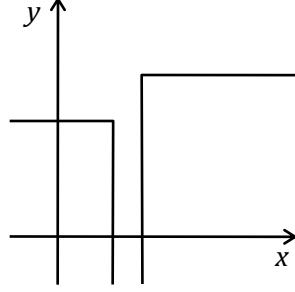


Figure 6: Stair-lines in the plane.

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A Generalization to k -flats: an incomplete argument

We define *stair- k -flats*, which are the stair-convex analogues of k -flats. Stair- k -flats in \mathbb{R}^d are defined inductively on k and d . A stair- k -flat is always topologically equivalent to a regular k -flat; this fact follows by induction, and it is inductively necessary for the definition itself.

Definition A.1: A *stair-0-flat* is a point, and a *stair- d -flat* in \mathbb{R}^d is \mathbb{R}^d . For $1 \leq k \leq d-1$, a *stair- k -flat* f in \mathbb{R}^d has one of these three forms:

- (“Horizontal”) $f = f' \times \{z\}$ for some stair- k -flat f' in \mathbb{R}^{d-1} and some $z \in \mathbb{R}$.
- (“Vertical”) $f = f' \times (-\infty, \infty)$ for some stair- $(k-1)$ -flat f' in \mathbb{R}^{d-1} .
- (“Diagonal”) Let f' be a stair- $(k-1)$ -flat in \mathbb{R}^{d-1} , and let f'' be a stair- k -flat in \mathbb{R}^{d-1} that contains f' . It follows by induction that f' is topologically equivalent to \mathbb{R}^{k-1} and f'' is topologically equivalent to \mathbb{R}^k ; thus, f' partitions f'' into two relatively closed *half-stair- k -flats* whose intersection equals f' . Let h be one of these halves. Then,

$$f = (f' \times (-\infty, z]) \cup (h \times \{z\}),$$

for some $z \in \mathbb{R}$.

See Figures 6, 7, and 8 for some examples of stair-lines and stair-planes.

In a diagonal stair-flat f , the part $f' \times (-\infty, z]$ is called the *vertical part* of f , and the part $h \times \{z\}$ is called the *horizontal part* of f .

Diagonal stair-flats are the most general ones; the other ones can be considered diagonal stair-flats for which either its horizontal or its vertical part has been moved to infinity in some direction.

Lemma A.2. (1) *Stair-flats are stair-convex.* (2) *Closed half-stair-flats are stair-convex.*

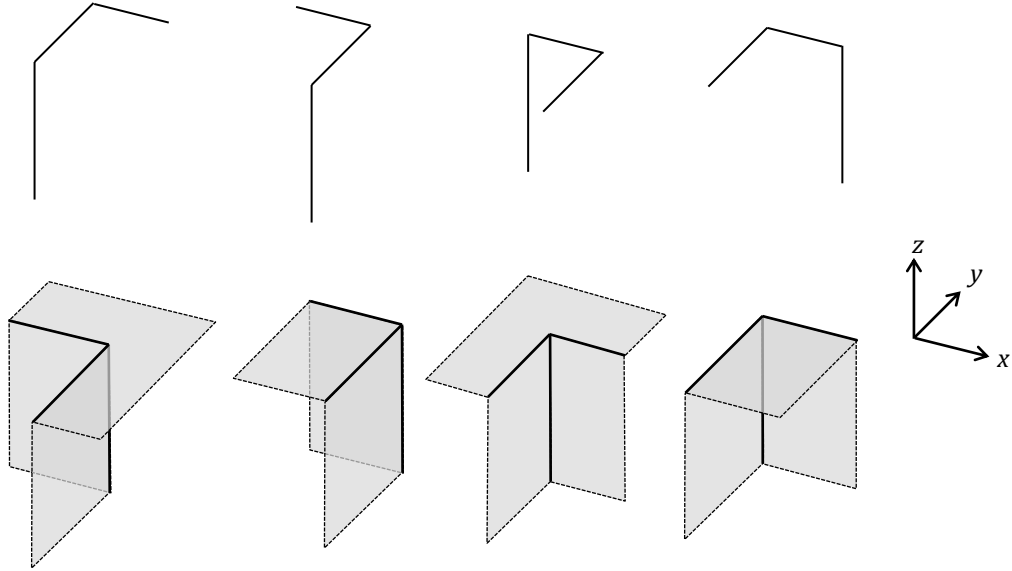


Figure 7: Stair-lines and stair-planes in \mathbb{R}^3 .

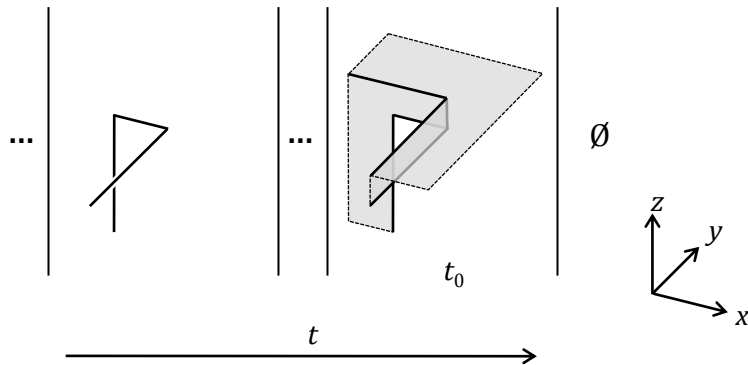


Figure 8: Example of a stair-plane in \mathbb{R}^4 . The fourth coordinate is t (“time”). From $t = -\infty$ up to a certain time t_0 there exists a static stair-line. At time t_0 a stair-halfplane bounded by this stair-line suddenly appears for an instant, and then everything disappears.

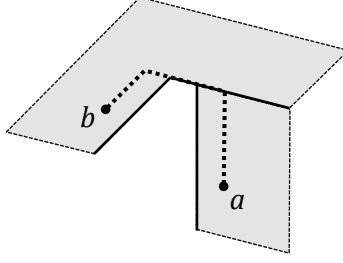


Figure 9: A closed half-stair-flat is always stair-convex, but an open one is not. The stair-path between a and b intersects the relative boundary of the pictured half-stairplane.

Proof. By induction. The first claim in \mathbb{R}^d easily follows from the second claim in \mathbb{R}^{d-1} and monotonicity of slices. And the second claim in \mathbb{R}^d follows from the first claim in \mathbb{R}^d as follows:

Let h be a closed half-stair- k -flat in \mathbb{R}^d . Its relative boundary is some stair- $(k-1)$ -flat f' . Let f be some stair- k -flat that contains h (note that f might not be unique). Let $a, b \in h$, and suppose for a contradiction that $\sigma = \sigma(a, b)$ is not completely contained in h . Since f is stair-convex, σ is completely contained in f . Thus, σ must cross f' in at least two points c and d when going from h to the other half of f . But f' is stair-convex, so the part of σ between c and d , which equals $\sigma(c, d)$, never leaves f' . Contradiction. \square

(We conjecture that every stair-convex set in \mathbb{R}^d that is topologically equivalent to \mathbb{R}^k is a actually stair- k -flat.)

Note that an *open* half-stair-flat, unlike a *closed* one, is not necessarily stair-convex; see Figure 9.

A.1 Equivalence to Euclidean flats

The following claim, for which we do not yet have a complete proof, states that indeed stair- k -flats are the stair-convex equivalent of Euclidean k -flats. Intuitively this means that, if m , the side of the stretched grid, is very large, and if f is a Euclidean k -flat that intersects BB , then $\pi(f \cap \text{BB})$ looks almost like a stair- k -flat intersected with $[0, 1]^d$. Conversely, every intersection of a stair- k -flat with $[0, 1]^d$ can be obtained this way.

Given two sets $f, g \subseteq \mathbb{R}^d$ and an integer $c \geq 0$, we say that f and g are c -close in BB if every point of $f \cap \text{BB}$ is c -close to a point of $g \cap \text{BB}$ and vice versa. Normally, f will be a flat and g will be a stair-flat.

Claim A.3. *For every Euclidean k -flat $f \subset \mathbb{R}^d$ there exists a stair- k -flat $g \subset \mathbb{R}^d$ that is c -close to f in BB for some constant $c = c(d)$, and vice versa.*

“Proof”. For the first direction, let f be a Euclidean k -flat. If f is vertical, then the desired stair- k -flat g is also vertical, and the claim follows by induction on d . So suppose f is not vertical.

Let \overline{BB} be the bottom face of BB , let $h_0 \subset \mathbb{R}^d$ be the horizontal hyperplane containing \overline{BB} , and let $f' = f \cap h_0$. If $f' = \emptyset$ then f is horizontal, so the desired stair- k -flat g is also horizontal, and it can again be constructed by induction on d .

Otherwise, f' is a $(k-1)$ -flat (which may or may not intersect \overline{BB}). By induction, f' is $c(d-1)$ -close to some stair- $(k-1)$ -flat g' in \overline{BB} .

Next, let \bar{f} be the vertical projection of f into h_0 . Again, by induction \bar{f} is $c(d-1)$ -close to some stair- k -flat g'' in \overline{BB} .

Furthermore, since $f' \subset \bar{f}$, we know that g' is very close to g'' .⁴

Now, let a and b be two points on $f \cap BB$, and let \bar{a} and \bar{b} be their vertical projections into \overline{BB} . One can verify that, if both \bar{a} and \bar{b} lie at Euclidean distance at least 1 from f' , then a and b are 1-close in last coordinate. (This follows by a simple calculation involving ratios; compare to Lemma 2.1.) Intuitively, in $\pi(f \cap BB)$ all points have almost the same height, except for those that are very close to $\pi(f' \cap BB)$ in the first $d-1$ coordinates, for which the height drops abruptly to zero.

Thus, the desired stair- k -flat g is obtained as follows: Let h be the “correct” half-stair-flat of g'' bounded by g' (the half corresponding to the half of f that goes up in last coordinate). Then pick an arbitrary point $a \in f$ with positive height such that its projection \bar{a} lies in \overline{BB} and has Euclidean distance greater than 1 from f' . Let $z = a_d$, and let

$$g = (g' \times (-\infty, z]) \cup (h \times \{z\}).$$

(The case where no such a exists can also be taken care of; we omit the details.)

Let us briefly sketch the other direction: Let

$$g = (g' \times (-\infty, z]) \cup (h \times \{z\}).$$

be a given stair- k -flat. Let $g'' \subset \mathbb{R}^{d-1}$ be a full stair- k -flat that contains h . By induction, let $f' \subset h_0$ be the Euclidean $(k-1)$ -flat that approximates g' in \overline{BB} , and let $f'' \subset h_0$ be the Euclidean k -flat that approximates g'' in \overline{BB} .

We know that f' is “close” to f'' —not in the Euclidean sense, but in the sense of c -closeness based on the mapping π . As before, somehow “snap” f' into f'' , getting a Euclidean $(k-1)$ -flat $f''' \subset f''$.

Let $\bar{a} \in f'''$ be a point that has Euclidean distance at least 1 from f''' ; elevate \bar{a} vertically to height z , getting point a ; and finally let f (the desired Euclidean k -flat) be the affine hull of f''' and a . \square

A.2 The generalized covering lemma

Lemma A.4 (Generalized covering lemma). *Let f be a stair- k -flat in \mathbb{R}^d . Then, there exists a family \mathcal{H} of $\binom{d-1}{k} \frac{d+k+1}{k+1}$ closed stair-halfspaces, each one containing f in its boundary, and together covering \mathbb{R}^d exactly $\binom{d-1}{k}$ times (apart from the points lying on the boundary of the stair-halfspaces of \mathcal{H} , which might be covered more times).*

⁴Here is the problem: We would like to have $g' \subset g''$. One solution would be to “snap” g' into g'' , whatever that means.

Proof. We construct \mathcal{H} by induction on k and d .

Let $\Gamma = \Gamma_{k,d} = \binom{d-1}{k} \frac{d+k+1}{k+1}$ denote the desired number of stair-halfspaces, and let $\Delta = \Delta_{k,d} = \binom{d-1}{k}$ denote the number of times space should be covered.

When $k = 0$, f consists of a single point a , and we have $\Gamma = d + 1$, $\Delta = 1$. In this case we let $H = \{C_0(a), \dots, C_d(a)\}$, and we have $d + 1$ stair-halfspaces, all containing a in their boundary, and together covering space exactly once, as required.

Now suppose $k \geq 1$, and assume that f is a diagonal stair- k -flat (the other types of stair-flats are degeneracies, as mentioned above). Thus, f has the form

$$f = (f' \times (-\infty, z]) \cup (h \times \{z\}),$$

for some stair- $(k - 1)$ -flat f' and some half-stair- k -flat h , both in \mathbb{R}^{d-1} , such that f' is the relative boundary of h . Let f'' be a full stair- k -flat in \mathbb{R}^{d-1} containing h (there might be more than one way to “complete” h into a stair-flat).

Let $\Gamma' = \Gamma_{k-1,d-1}$, $\Delta' = \Delta_{k-1,d-1}$, $\Gamma'' = \Gamma_{k,d-1}$, $\Delta'' = \Delta_{k,d-1}$. By induction, we can construct a family \mathcal{H}' of Γ' stair-halfspaces in \mathbb{R}^{d-1} , all containing f' in their boundary and covering \mathbb{R}^{d-1} exactly Δ' times, and a family \mathcal{H}'' of Γ'' stair-halfspaces in \mathbb{R}^{d-1} , all containing f'' in their boundary and covering \mathbb{R}^{d-1} exactly Δ'' times.

Also note the following identities:

$$\begin{aligned} \Gamma &= \Gamma' + \Gamma'', \\ \Delta &= \Delta' + \Delta'', \\ \Gamma' &= \Delta + \Delta'. \end{aligned} \tag{4}$$

We will construct our desired family \mathcal{H} of stair-halfspaces as $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$, with $|\mathcal{H}_1| = |\mathcal{H}'| = \Gamma'$ and $|\mathcal{H}_2| = |\mathcal{H}''| = \Gamma''$.

Let us start by constructing \mathcal{H}_2 , which is easier. We let

$$\mathcal{H}_2 = \{H'' \times (-\infty, z] : H'' \in \mathcal{H}''\}$$

(namely, we “extrude” each halfspace of \mathcal{H}'' in the d -th direction from $-\infty$ to z).

Let $H = H'' \times (-\infty, z]$ be a stair-halfspace in \mathcal{H}_2 . Note that H contains all of f , as required. Furthermore, the boundary of H is

$$\partial H = (\partial H'' \times (-\infty, z]) \cup (H'' \times \{z\}),$$

so f is actually contained in the boundary of H , as required.

Note that the stair-halfspaces of \mathcal{H}_2 cover the “lower part” of \mathbb{R}^d (meaning, $\mathbb{R}^{d-1} \times (-\infty, z]$) exactly Δ'' times, and they do not cover the “upper part” of \mathbb{R}^d (meaning, $\mathbb{R}^{d-1} \times [z, \infty)$) at all.

We now construct \mathcal{H}_1 . Let us first take a more careful look at the stair-halfspaces of \mathcal{H}' . Recall that the stair-halfspaces of \mathcal{H}' were only “designed” to contain f' , but not h .

Lemma A.5. *The family \mathcal{H}' can be partitioned into $\mathcal{H}' = \mathcal{H}'_a \cup \mathcal{H}'_b \cup \mathcal{H}'_c$, such that:*

- h intersects the interior of every $H \in \mathcal{H}'_a$;

- h is contained in the boundary of every $H \in \mathcal{H}'_b$; and
- h is not contained in any $H \in \mathcal{H}'_c$.

Proof. We have to show that, if h intersects the interior of some $H \in \mathcal{H}'$, then $h \subset H$. Recall that the relative boundary of h , namely f' , lies on the *boundary* of every $H \in \mathcal{H}'$.

So suppose for a contradiction that there exists a half-stair- k -flat $h \subset \mathbb{R}^d$ and there exists a stair-halfspace $H \subset \mathbb{R}^d$ such that ∂H contains the relative boundary of h , and such that h intersects *both* the interior of H and $\mathbb{R}^d \setminus H$. Then, by Claim A.3 we could construct a similar configuration with a *Euclidean* half- k -flat and a *Euclidean* halfspace. But that is clearly impossible.⁵ \square

Claim A.6. *We have $|\mathcal{H}'_a| \leq \Delta' \leq |\mathcal{H}'_a \cup \mathcal{H}'_b|$ and $|\mathcal{H}'_c| \leq \Delta \leq |\mathcal{H}'_b \cup \mathcal{H}'_c|$.*

“Proof”. Let $S = \mathbb{R}^d \setminus \bigcup_{H \in \mathcal{H}'} \partial H$ be the set of all points not lying on the boundary of any stair-halfspace of \mathcal{H}' .

Suppose first that our half-stair-flat h intersects S (so h is, in some sense, “generic”), and let a be a point in $h \cap S$. Then, by Lemma A.5, for every $H \in \mathcal{H}'$ we have $h \subset H$ if and only if $a \in H$. By the construction of \mathcal{H}' we know that it contains exactly Δ' halfspaces that satisfy this latter property; therefore, $|\mathcal{H}'_a| = \Delta'$ and $\mathcal{H}'_b = \emptyset$. Then equation (4) implies that $|\mathcal{H}'_c| = \Delta$, and we are done.

If h does not intersect S , then we apply a limit argument: h is arbitrarily close to a half-stair- k -flat h' , having the same relative boundary as h , such that h' *does* intersect S .⁶ Define the sets $\mathcal{H}'_a, \mathcal{H}'_b, \mathcal{H}'_c$ of Lemma A.5 for h' , and continuously “rotate” h' until it matches h . At the beginning, we have $|\mathcal{H}'_a| = \Delta'$, $\mathcal{H}'_b = \emptyset$; and the only combinatorial changes that can occur involve moving *into* the boundary of stair-flats $H \in \mathcal{H}'$. In other words, elements can only move from \mathcal{H}'_a or \mathcal{H}'_c into \mathcal{H}'_b . \square

Now, partition \mathcal{H}' into $\mathcal{H}' = \mathcal{H}'_{\text{up}} \cup \mathcal{H}'_{\text{down}}$ such that $\mathcal{H}'_c \subseteq \mathcal{H}'_{\text{up}}$ and $\mathcal{H}'_a \subseteq \mathcal{H}'_{\text{down}}$, and such that $|\mathcal{H}'_{\text{up}}| = \Delta$ and $|\mathcal{H}'_{\text{down}}| = \Delta'$.

Then our desired family \mathcal{H}_1 is

$$\begin{aligned} \mathcal{H}_1 = & \{ (H \times (-\infty, z]) \cup (\mathbb{R}^{d-1} \times [z, \infty)) : H \in \mathcal{H}'_{\text{up}} \} \\ & \cup \{ H \times (-\infty, z] : H \in \mathcal{H}'_{\text{down}} \} \end{aligned}$$

(extruding every stair-halfspace in the d -th direction as before, and adding the d -th component only to the stair-halfspaces of \mathcal{H}'_{up}).

The condition $\mathcal{H}'_c \subseteq \mathcal{H}'_{\text{up}}$ implies that every stair-halfspace of \mathcal{H}_1 contains $h \times \{z\}$, and thus all of f .

Moreover, for a stair-halfspace $H \in \mathcal{H}'_{\text{up}}$, the boundary of the corresponding stair-halfspace $H' \in \mathcal{H}_1$ is

$$\partial H' = (\partial H \times (-\infty, z]) \cup ((\mathbb{R}^{d-1} \setminus H) \times \{z\}),$$

⁵We would like a proof of Lemma A.5 that uses only the notions of stair-convexity and does not invoke Claim A.3; but we have not found such a proof.

⁶This would need a proof, of course.

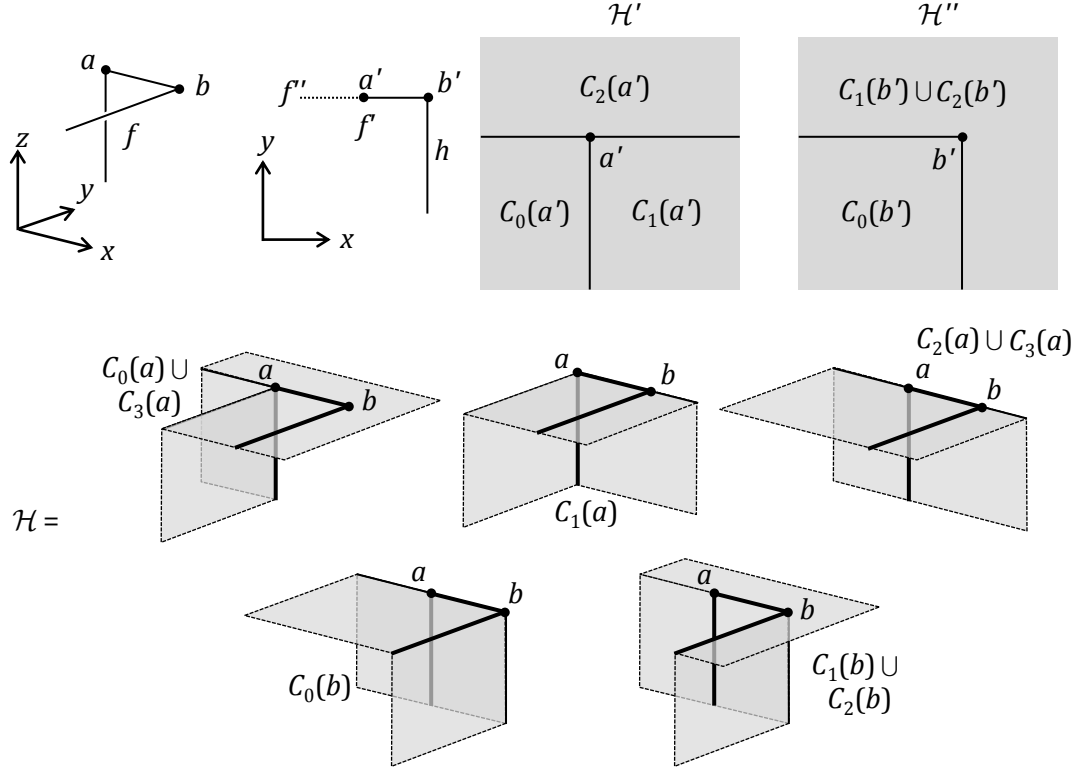


Figure 10: An example of constructing \mathcal{H} from \mathcal{H}' and \mathcal{H}'' .

so actually $f \in \partial H'$.

Finally, note that \mathcal{H}_1 covers the “lower part” of \mathbb{R}^d exactly Δ' times and the “upper part” of \mathbb{R}^d exactly Δ times. Hence, $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ is our desired family. \square

Figure 10 shows an example of the construction of Lemma A.4 for a stair-line f (so $k = 1$) in \mathbb{R}^3 . The stair-line f is shown at the top left. To its right are shown the two-dimensional components from which f is made: The stair-point f' and the stair-ray h . The stair-line f'' that contains h is also shown.

Next are shown the three stair-halfplanes of \mathcal{H}' , which contain f' in their boundary and together cover the plane exactly once. We have $\mathcal{H}' = \{C_0(a'), C_1(a'), C_2(a')\}$.

Next are the two stair-halfplanes of \mathcal{H}'' , which contain f'' in their boundary and together cover the plane exactly once. We have $\mathcal{H}'' = \{C_0(b'), C_1(b') \cup C_2(b')\}$.

Finally are shown the five stair-halfspaces of the desired set \mathcal{H} , which contain f in their boundary and together cover space exactly twice. We have

$$\mathcal{H} = \{C_0(a) \cup C_3(a), C_1(a), C_2(a) \cup C_3(a), C_0(b), C_1(b) \cup C_2(b)\}.$$

The adventurous reader might want to try to construct the seven stair-halfspaces that cover \mathbb{R}^4 three times for the stair-plane in Figure 8.